Chem221a : Solution Set 3

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Problem 1

It's important to remember that operators are defined as equal if they have the same effect on an arbitrary ket. So, although we don't need the ket to get the "right answer," we haven't done the problem correctly unless we include a ket (which I'll call $|\psi\rangle$) for the problems in which the answers are operators.

(a) We know that $(U(m,n)\,|\psi\rangle)^\dagger = \langle\psi|\,U^\dagger(m,n)$. So we have:

$$\begin{aligned}
\langle\psi|\,U^\dagger(m,n) &= (U(m,n)\,|\psi\rangle)^\dagger && (1)\\
&= (|\phi_m\rangle\,\langle\phi_n|\psi\rangle)^\dagger && (2)\\
&= \langle\phi_m|\,\langle\phi_n|\psi\rangle^* && (3)\\
&= \langle\psi|\phi_n\rangle\,\langle\phi_m| && (4)\\
&= \langle\psi|\,U(n,m) && (5)
\end{aligned}$$

So have determined that $U^\dagger(m,n) = U(n,m)$.

(b) Now we calculate the commutator. As always when we calculate a commutator, we operate it on an arbitrary state (in this case, $|\psi\rangle$).

$$\begin{aligned}
[H, U(m,n)]\,|\psi\rangle &= HU(m,n)\,|\psi\rangle - U(m,n)H\,|\psi\rangle && (6)\\
&= H\,|\phi_m\rangle\,\langle\phi_n|\psi\rangle - |\phi_m\rangle\,\langle\phi_n|H|\psi\rangle && (7)
\end{aligned}$$

But the $|\phi_n\rangle$ are eigenstates of the operator H (we'll say they have eigenvalue E_n, keeping with the Hamiltonian/energy eigestates notation). This means that $H\,|\phi_n\rangle = E_n\,|\phi_n\rangle$ and $\langle\phi_n|\,H = E_n\,\langle\phi_n|$. Therefore we have

$$\begin{aligned}
[H, U(m,n)]\,|\psi\rangle &= E_m\,|\phi_m\rangle\,\langle\phi_n|\psi\rangle - E_n\,|\phi_m\rangle\,\langle\phi_n|\psi\rangle && (8)\\
&= (E_m - E_n)\,|\phi_m\rangle\,\langle\phi_n|\psi\rangle && (9)\\
&= (E_m - E_n)U(m,n)\,|\psi\rangle && (10)
\end{aligned}$$

from which we determine that $[H, U(m,n)] = (E_m - E_n)U(m,n)$.

(c)

$$U(m,n)U^\dagger(p,q)\,|\psi\rangle = U(m,n)U(q,p)\,|\psi\rangle \qquad = |\phi_m\rangle\,\langle\phi_n|\phi_q\rangle\,\langle\phi_p|\psi\rangle \qquad (11)$$

where I have used the result from part (a). Now, remembering that the braket $\langle\phi_n|\phi_q\rangle$ will be just a number, we commute it over to the side:

$$U(m,n)U^\dagger(p,q)\,|\psi\rangle = \langle\phi_n|\phi_q\rangle\,|\phi_m\rangle\,\langle\phi_p|\psi\rangle \tag{12}$$

$$= \langle\phi_n|\phi_q\rangle\,U(m,p)\,|\psi\rangle \tag{13}$$

Now we'll use the fact that the $|\phi_n\rangle$ form an orthonormal basis. Although the problem says we're allowed to assume this, you should remember that this can be proven for any Hermitian operator H. This means that the overlap of $|\phi_n\rangle$ and $|\phi_q\rangle$ is zero if the states are different (*i.e.*, the states are othogonal) or one if the states are the same (*i.e.*, the states are normalized). Combined, these conditions are orthonormality, and hence we have $\langle\phi_n|\phi_q\rangle = \delta_{n,q}$, a Kronecker delta. This gives us our goal:

$$U(m,n)U^\dagger(p,q) = \delta_{n,q}U(m,p) \tag{14}$$

(d) To calculate the trace, we sum over all the diagonal matrix elements. Mathematically, we want to determine

$$\mathrm{Tr}(U(m,n)) = \sum_i \langle\phi_i|U(m,n)|\phi_i\rangle \tag{15}$$

$$= \sum_i \langle\phi_i|\phi_m\rangle\,\langle\phi_n|\phi_i\rangle \tag{16}$$

$$= \sum_i \langle\phi_n|\phi_i\rangle\,\langle\phi_i|\phi_m\rangle \tag{17}$$

$$= \langle\phi_n|\left(\sum_i |\phi_i\rangle\,\langle\phi_i|\right)|\phi_m\rangle \tag{18}$$

So far, we have only used the commutative and associative properties of complex multiplication. Now we use the fact that the $|\phi_i\rangle$ form a complete basis to take out a resolution of the identiy, and then we use the orthonormality of the eigenstates to get our solution:

$$\mathrm{Tr}(U(m,n)) = \langle\phi_n|1|\phi_m\rangle \tag{19}$$

$$= \langle\phi_n|\phi_m\rangle = \delta_{n,m} \tag{20}$$

(e) In the previous part, we took out a resolution of the identity; in this part, we'll insert two:

$$A\,|\psi\rangle = \left(\sum_m |\phi_m\rangle\,\langle\phi_m|\right) A \left(\sum_n |\phi_n\rangle\,\langle\phi_n|\right)|\psi\rangle \tag{21}$$

$$= \sum_{m,n} |\phi_m\rangle\,\langle\phi_m|A|\phi_n\rangle\,\langle\phi_n|\psi\rangle \tag{22}$$

$$= \sum_{m,n} \langle\phi_m|A|\phi_n\rangle\,|\phi_m\rangle\,\langle\phi_n|\psi\rangle \tag{23}$$

$$= \sum_{m,n} A_{mn}U(m,n)\,|\psi\rangle \tag{24}$$

Since the two sides do the same thing to any arbitrary ket $|\psi\rangle$, we've shown that they are equal.

(f) For the final part of this problem, we'll again insert a resolution of the identity, but then we'll use the fact that anything enclosed by a bra and a ket makes a number to use commutativity to our advantage.

$$\begin{aligned}
A_{pq} &= \langle\phi_p|A|\phi_q\rangle && (25)\\
&= \langle\phi_p| \left(\sum_i |\phi_i\rangle \langle\phi_i|\right) A\,|\phi_q\rangle && (26)\\
&= \sum_i \langle\phi_p|\phi_i\rangle \langle\phi_i|A|\phi_q\rangle && (27)\\
&= \sum_i \langle\phi_i|A|\phi_q\rangle \langle\phi_p|\phi_i\rangle && (28)\\
&= \sum_i \langle\phi_i|\,(A\,|\phi_q\rangle \langle\phi_p|)\,|\phi_i\rangle && (29)\\
&= \sum_i \langle\phi_i|\,(AU(q,p))\,|\phi_i\rangle && (30)\\
&= \mathrm{Tr}(AU^\dagger(p,q)) && (31)
\end{aligned}$$

where the last step has used the result from part (a) as well as the definition of the trace.

Problem 2

(a) There are actually a couple of ways to go about this problem. One is to follow the hint from the problem set. Another is to show that all operators have at least one eigenvalue (I would let you state this as an assumption — I'll give the proof in a separate document). From that fact, it is rather straighforward to show that the only possible eigenvalues are 1 or 0.

(i) To get the first part of this problem (the eigenvalues), it's probably easiest just to assume that there exists an eigenket $|a\rangle \neq 0$ of the operator P with some eigenvalue a. Since $P = P^2$, we have:

$$\begin{aligned}
P\,|a\rangle &= P^2\,|a\rangle && (32)\\
a\,|a\rangle &= a^2\,|a\rangle && (33)\\
(a-a^2)\,|a\rangle &= 0 && (34)
\end{aligned}$$

where this last equation is clearly satisfied only if $a - a^2 = 0$ or $|a\rangle = 0$. The second contradicts our assumption,[1] so the first statement must be true. But the factorization $a - a^2 = a(1-a)$ tells us that the eigenvalue a may only take on the values $a = 1$ or $a = 0$. □

Unfortunately, this method doesn't do much to help us identify the eigenkets of P, so the next method is probably more useful.

[1] I remind you that this assumption can be proven — see additional files on the website for the proof.

(ii) If we don't make that assumption, we can (with a little bit fancier footwork) find the same result from the relation $P = P^2$ on an arbitrary ket:

$$P |\alpha\rangle = P^2 |\alpha\rangle \tag{35}$$
$$= P(P |\alpha\rangle) \tag{36}$$

If we call $|\beta\rangle = P |\alpha\rangle$, this equation becomes

$$1 |\beta\rangle = P |\beta\rangle \tag{37}$$

which is obviously an eigenvalue equation stating that $|\beta\rangle = P |\alpha\rangle$ is an eigenket with eigenvalue 1.

To get the eigenvalue of zero, we use the hint from the problem set. Specifically, we look at the operator $P(1 - P)$. Using this on an arbitrary ket α, we obtain

$$P(1 - P) |\alpha\rangle = P |\alpha\rangle - P^2 |\alpha\rangle \tag{38}$$

Remembering that $P^2 = P$, this immediately gives us the result:

$$P(1 - P) |\alpha\rangle = 0 \tag{39}$$

Again, since on operator acting on a ket gives us another ket, we can define the ket $|\beta'\rangle = (1 - P) |\alpha\rangle$, and the equation becomes

$$P |\beta\rangle = 0 |\beta\rangle \tag{40}$$

stating that $|\beta\rangle$ is an eigenket of P with eigenvalue 0.

With this method, it is a little harder to show that these are the *only* possible eigenvalues. The trick is to recognize that we have defined two operators which project into the eigenspaces for each eigenvalue. First we showed that the operator P projects the arbitrary ket $|\alpha\rangle$ into the set of kets $|\beta\rangle$ which are eigenkets with eigenvalue 1. Then we showed that $1 - P$ projects into the set of kets $|\beta'\rangle$ which are eigenkets with eigenvalue 0. The trick is to take the sum of these operators: $(1 - P) + (P) = 1$, the identity operator. So the direct sum of the two eigenspaces (which are orthogonal) gives the whole space. That means that we have found all the eigenvalues.

If this is confusing you, imagine projectors in 3D space. Let $P = P_{xy}$, the projector into the xy-plane. For any 3D vector $|\alpha\rangle = \alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z}$, we have $P_{xy} |\alpha\rangle = \alpha_x \hat{x} + \alpha_y \hat{y}$. The operator $1 - P = P_z$, the projector onto the z-axis. So, in this case, $|\beta\rangle$ is any vector in the xy-plane, and $|\beta'\rangle$ is any vector along the z-axis. It is worth remembering that there is actually a set of vectors $\{|\beta\rangle\}$ and $\{|\beta'\rangle\}$, not just one of each. And, as we see in this example, the sum $(1 - P) + (P)$ gives us the identity operator, which means that any vector in the space can be decomposed in terms of vectors in the two eigenspaces. Therefore, we have found all the possible eigenvalues.

Once we've used the second method, it is easy to identify the eigenkets: obviously, $|\beta\rangle = P |\alpha\rangle$ for any arbitrary $|\alpha\rangle$ will be an eigenket of P with eigenvalue 1. Similarly, $|\beta'\rangle = (1 - P) |\alpha\rangle$ will be an eignket of P with eigenvalue 0 for any ket $|\alpha\rangle$.

(b) We can just accept that this operator does the projection described (although it should be easy for you to see that this is true). All we need to do is show that $P_q^2 = P_q$. Again, when dealing with operators, we need to show the equality by operating it on an arbitrary ket, $|\psi\rangle$.

$$P_q^2 |\psi\rangle = \left(\sum_{i=1}^{q} |a_i'\rangle \langle a_i'|\right) \left(\sum_{j=1}^{q} |a_j'\rangle \langle a_j'|\right) |\psi\rangle \tag{41}$$

$$= \sum_{i=1}^{q} \sum_{j=1}^{q} |a_i'\rangle \langle a_i'|a_j'\rangle \langle a_j'|\psi\rangle \tag{42}$$

$$= \sum_{i=1}^{q} \sum_{j=1}^{q} |a_i'\rangle \delta_{i,j} \langle a_j'|\psi\rangle \tag{43}$$

Now summing over j will cause the Kronecker delta to set $i = j$, so we have

$$P_q^2 |\psi\rangle = \left(\sum_{i=1}^{q} |a_i'\rangle \langle a_i'|\right) |\psi\rangle \tag{44}$$

$$= P_q |\psi\rangle \tag{45}$$

Hence we have $P_q^2 = P_q$.

Problem 3

As Shankar says, this problem is "very important." The hope is that this exercise will help you make connections between mathematical representation of quantum mechanics and its physical interpretation.

For what it's worth, the operators discussed in this problem are representations of the 3D angular momentum operators in the L_z basis. We'll come across these operators again when we talk about angular momentum.

(a) The possible values one can obtain for an operator are its eigenvalues (that's a postulate of quantum mechanics) and a diagonal matrix has its eigenvalues on the diagonal (that's a fundamental result of linear algebra). Putting those two facts together and looking at the matrix representation of L_z, we immediately see that the possible values of L_z are $+1, 0, -1$.

(b) First, we need to define the state where $L_z = 1$. As we noticed before, 1 is a possible value for the operator L_z (otherwise this question wouldn't make sense) because it is on the diagonal of the diagonalized representation of the L_z operator. In that representation, the eigenvalue 1 is the element $(L_z)_{11}$ — the element in the first columnand first row. This means that the eigenvector associated with it (in that representation) is the first canonical basis vector, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Now that we have a representation of the state, we can calculate the quantities we're asked for. Expectation values are calculated by putting the bra and ket versions of the state on either side of the operator:

$$\langle L_x \rangle_{L_z=1} = \langle \mathbf{e}_1 | L_x | \mathbf{e}_1 \rangle \tag{46}$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{47}$$

$$= 0 \tag{48}$$

You might notice that the $\langle \mathbf{e_1}|$ and $|\mathbf{e_1}\rangle$ have the effect of picking out the 1,1 element of the matrix in question. The canonical vectors will always do this, and if you've chosen your representation such that you're in a canonical basis, that can be a very useful (and time-saving) trick.

The matrix representation of the L_x^2 operator is obtained by simply taking the square of the matrix representation of L_x:

$$\langle L_x^2 \rangle_{L_z=1} = \langle \mathbf{e}_1 | L_x^2 | \mathbf{e}_1 \rangle \tag{49}$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{50}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{51}$$

$$= \frac{1}{2} \tag{52}$$

Finally, we calculate ΔL_x:

$$\Delta L_x = \langle L_x^2 \rangle - \langle L_x \rangle^2 \tag{53}$$

$$= \frac{1}{2} - 0^2 = \frac{1}{2} \tag{54}$$

(c) One thing to clear up is that all of these operators are represented in the L_z basis. We know that because the representation of L_z is diagonal in the L_z basis (as is the representation from the problem). We just assume that all the operators are represented in the same basis — if that wasn't the case, we'd just say that Shankar is evil incarnate and give up.

First we'll calculate the eigenvalues of the L_x operator, which will be the same in any representation:

$$0 = \det \left(\lambda - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \tag{55}$$

$$= \det \begin{pmatrix} \lambda & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & \lambda & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & \lambda \end{pmatrix} \tag{56}$$

$$= \lambda^3 - \frac{1}{2}\lambda - \frac{1}{2}\lambda \tag{57}$$

$$= \lambda(\lambda - 1)(\lambda + 1) \tag{58}$$

So our eigenvalues are 0, +1, and −1 (as we'd expect).

To calculate the associated eigenvectors, we plug each eigenvalue in for λ in the eigenequation. Since we have three different eigenvalues, each is associated with a single eigenvector. That means that each matrix below will be of rank 2: two of the equations will be linearly independent, and the third will be some combination of the other two. We'll use that fact to determine the relative values of the components of the eigenvectors. Normalization will make the eigenvectors unique.

Eigenvector for $\lambda = +1$:

$$\begin{pmatrix} 1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{59}$$

By inspection, we see that the equation given by the second row of the matrix is equal to the sum of the equations from the sums of the first and third row equations, multiplied by $-1/\sqrt{2}$. So we'll use the first and third rows as the equations to obtain our eigenvector.

The first row equation is $x_1 - x_2/\sqrt{2} = 0$, which gives us $x_2 = \sqrt{2}x_1$. Similarly, the third row equation gives us $x_2 = \sqrt{2}x_3$. Together, these show that $x_1 = x_3$. If we set $x_1 = 1$, then we have $x_3 = 1$ and $x_2 = \sqrt{2}$. We'll use those as the components $|v_+\rangle$, and the normalized version of that vector will be our state $|L_x = +1\rangle$:

$$|L_x = +1\rangle = \frac{1}{\sqrt{\langle v_+|v_+\rangle}}\,|v_+\rangle \tag{60}$$

$$= \frac{1}{\sqrt{1^2 + \sqrt{2}^2 + 1^2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \tag{61}$$

Eigenvector for $\lambda = 0$:

$$\begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{62}$$

In this case, the first row equation and third row equation are obviously identical, and tell that the $x_2 = 0$. The second row equation tells us that $x_1 = -x_3$. Taking $x_1 = 1$, we get $|v_0\rangle$ (shown below) as the unnormalized vector. We normalize is normally:

$$|L_x = 0\rangle = \frac{1}{\sqrt{\langle v_0|v_0\rangle}}\,|v_0\rangle \tag{63}$$

$$= \frac{1}{\sqrt{1^2 + 0^2 + (-1)^2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \tag{64}$$

Eigenvector for $\lambda = -1$

$$\begin{pmatrix} -1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{65}$$

The analysis is pretty much the same here as it was for the $_1$ eigenvalue, just with a sign difference between first/third row equations and the second row equation. We obtain the normalized eigenket:

$$|L_x = 0\rangle = \frac{1}{\sqrt{\langle v_0|v_0\rangle}}|v_0\rangle \tag{66}$$

$$= \frac{1}{\sqrt{1^2 + (-\sqrt{2})^2 + 1^2}}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/\sqrt{2}\\1/2\end{pmatrix} \tag{67}$$

It's worth noting that these vectors (which are represented in the L_z basis) are orthonormal. You can show this yourself.

(d) Again, we have to define our state first. Reasoning similar to what we used in part (b) will suggest that we should represent this state as $|L_z = -1\rangle \doteq \mathbf{e_3} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. In a previous part of this problem, we defined the representations in the L_z basis for the $|L_x = i\rangle$ states, with $i \in \{1, 0, -1\}$.

The probability that a system in the state $|L_z = -1\rangle$ will be measured in a state $|L_x = i\rangle$ is, as usual, given by $|\langle L_x = i|L_z = -1\rangle|^2$. So let's do that for each possible value of L_x:

$$P(L_x = +1) = |\langle L_x = +1|L_z = -1\rangle|^2 \tag{68}$$

$$= \left|\begin{pmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4} \tag{69}$$

$$P(L_x = 0) = |\langle L_x = 0|L_z = -1\rangle|^2 \tag{70}$$

$$= \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{2} \tag{71}$$

$$P(L_x = -1) = |\langle L_x = -1|L_z = -1\rangle|^2 \tag{72}$$

$$= \left|\begin{pmatrix}\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4} \tag{73}$$

Note that the sum of these probabilities is 1, meaning that we have all the possibilities (and that our states are normalized).

(e) This time, we're given a state. All we have to do is interpret what it means. First we need to look at the state L_z^2. Squaring the representation of the L_z operator in the L_z basis, we obtain:

$$L_z^2 = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} \tag{74}$$

We still have a diagonal matrix, so the eigenstates of L_z are also eigenstates of L_z^2.[2] There are two canonical vectors which give us $L_z^2 = 1$: either $\mathbf{e_1}$ or $\mathbf{e_3}$. So we add up the probabilities of each:

$$P(L_z^2 = 1) = |\langle \mathbf{e_1}|\psi\rangle|^2 + |\langle \mathbf{e_3}|\psi\rangle|^2 \tag{75}$$

$$= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{3}{4} \tag{76}$$

This is the probability of this result. Now we answer the previous question, which is the state after the measurement. To get that, first we project our original state onto the possible states with $L_z^2 = 1$, then we renormalize the state. We'll call the unnormalized version of the state $|\psi'\rangle$ and the normalized version $\left|L_z^2 = 1\right\rangle$. So that gives us:

$$|\psi'\rangle = |\mathbf{e_1}\rangle \langle \mathbf{e_1}|\psi\rangle + |\mathbf{e_3}\rangle \langle \mathbf{e_3}|\psi\rangle \tag{77}$$

$$= \frac{1}{2} |\mathbf{e_1}\rangle + \frac{1}{\sqrt{2}} |\mathbf{e_3}\rangle \tag{78}$$

$$= \begin{pmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \tag{79}$$

Now we normalize that:

$$\left|L_z^2 = 1\right\rangle = \frac{1}{\sqrt{\langle\psi'|\psi'\rangle}} |\psi'\rangle \tag{80}$$

$$= \frac{1}{\frac{1}{2}^2 + 0^2 + \frac{1}{\sqrt{2}}^2} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{81}$$

$$= \frac{2}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{82}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} \tag{83}$$

That's the state we have *after* measuring L_z^2. To find the probability of each result from measuring L_z in this state, we simply take the inner product of this state vector with the state vector representing the eigenvalue of L_z. As above, we saw that $|L_z = 1\rangle = |\mathbf{e_1}\rangle$, $|L_z = 0\rangle =$

[2] In general, if an operator $\hat{B}$ can written as a linear function of another operator $\hat{A}$, there will exist a set of kets which are eigenkets for both operators.

$|\mathbf{e_2}\rangle$, and $|L_z = -1\rangle = |\mathbf{e_3}\rangle$. So,

$$P(L_z = 1) = \left|\langle \mathbf{e_1}\big| |L_z^2 = 1\rangle\rangle\right|^2 \tag{84}$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3} \tag{85}$$

$$P(L_z = 0) = \left|\langle \mathbf{e_2}\big| |L_z^2 = 1\rangle\rangle\right|^2 = 0 \tag{86}$$

$$P(L_z = -1) = \left|\langle \mathbf{e_3}\big| |L_z^2 = 1\rangle\rangle\right|^2 \tag{87}$$

$$= \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3} \tag{88}$$

So we can only get $L_z = 1$, with probability $\frac{1}{3}$, or $L_z = -1$, with probability $\frac{2}{3}$.

I hope that by the end of this class, you'll be able to do these calculations pretty much in your head.

(f) Let me start by trying to convince you of what you were supposed to convince yourself. We know that a general state can be written as

$$|\psi\rangle = c_+ |L_z = 1\rangle + c_0 |L_z = 0\rangle + c_- |L_z = -1\rangle \tag{89}$$

where the coefficients are, in general, complex. We also know the probabilities of each state are

$$P(L_z = 1) = \frac{1}{4} = |c_+|^2 \tag{90}$$

$$P(L_z = 0) = \frac{1}{2} = |c_0|^2 \tag{91}$$

$$P(L_z = -1) = \frac{1}{4} = |c_-|^2 \tag{92}$$

Of course, any complex number can be written as $z = |z|\, e^{i\delta_z}$, so that gives us

$$c_+ = \frac{e^{i\delta_1}}{2} \tag{93}$$

$$c_0 = \frac{e^{i\delta_2}}{\sqrt{2}} \tag{94}$$

$$c_- = \frac{e^{i\delta_3}}{2} \tag{95}$$

where δ_1, δ_2, δ_3 are undetermined. Are you convinced? I am!

So let's see whether those phase factors are completely irrelevant. The problem suggests that we try it out by calculating $P(L_x = 0)$. We do that by calculating the modulus square of the overlap of the state we're given with the state $|L_x = 0\rangle$, which we determined above:

$$|\langle L_x = 0|\psi\rangle|^2 = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\delta_1}}{2} \\ \frac{e^{i\delta_2}}{\sqrt{2}} \\ \frac{e^{i\delta_3}}{2} \end{pmatrix} \right|^2 \tag{96}$$

$$= \left| \frac{1}{2\sqrt{2}} e^{i\delta_1} + \frac{1}{2\sqrt{2}} e^{i\delta_3} \right|^2 \tag{97}$$

Remembering that the modulus square is given by the number times its complex conjugate, we continue:

$$|\langle L_x = 0|\psi\rangle|^2 = \left(\frac{1}{2\sqrt{2}} e^{i\delta_1} + \frac{1}{2\sqrt{2}} e^{i\delta_3} \right) \left(\frac{1}{2\sqrt{2}} e^{-i\delta_1} + \frac{1}{2\sqrt{2}} e^{-i\delta_3} \right) \tag{98}$$

$$= \frac{1}{8} \left(2 + e^{i(\delta_1 - \delta_3)} + e^{-i(\delta_1 - \delta_3)} \right) \tag{99}$$

As many of you will notice, this is going to turn into a cosine of the difference of the deltas. But we don't need to take it there to see the point: this is clearly dependent on those deltas. More specifically, it depends on the difference of those deltas. So the deltas matter not in the absolute, but in a relative way. The fact that the phase factors can only be measured in difference is what allows us to choose phase conventions. However, the phase factors are certainly not irrelevant.

Problem 4

The first step in this problem is to translate the sum into a matrix. We recall that the matrix elements will be given as:

$$\hat{H} = \begin{pmatrix} \left\langle 1 \middle| \hat{H} \middle| 1 \right\rangle & \left\langle 1 \middle| \hat{H} \middle| 2 \right\rangle \\ \left\langle 2 \middle| \hat{H} \middle| 1 \right\rangle & \left\langle 2 \middle| \hat{H} \middle| 2 \right\rangle \end{pmatrix} \tag{100}$$

Calculating each of these, we obtain:

$$\left\langle 1 \middle| \hat{H} \middle| 1 \right\rangle = \langle 1| \left(a \left(|1\rangle \langle 1| - |2\rangle \langle 2| + |1\rangle \langle 2| + |2\rangle \langle 1| \right) \right) |1\rangle = a \tag{101}$$

$$\left\langle 1 \middle| \hat{H} \middle| 2 \right\rangle = \langle 1| \left(a \left(|1\rangle \langle 1| - |2\rangle \langle 2| + |1\rangle \langle 2| + |2\rangle \langle 1| \right) \right) |2\rangle = a \tag{102}$$

$$\left\langle 2 \middle| \hat{H} \middle| 1 \right\rangle = \langle 2| \left(a \left(|1\rangle \langle 1| - |2\rangle \langle 2| + |1\rangle \langle 2| + |2\rangle \langle 1| \right) \right) |1\rangle = a \tag{103}$$

$$\left\langle 2 \middle| \hat{H} \middle| 2 \right\rangle = \langle 2| \left(a \left(|1\rangle \langle 1| - |2\rangle \langle 2| + |1\rangle \langle 2| + |2\rangle \langle 1| \right) \right) |2\rangle = -a \tag{104}$$

That gives us the matrix representation of $\hat{H}$:

$$\hat{H} = a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{105}$$

As always, we start by finding the eigenvalues.

$$
\begin{aligned}
0 &= \det\left(\begin{pmatrix} \lambda - a & -a \\ -a & \lambda + a \end{pmatrix}\right) && (106)\\
&= (\lambda - a)(\lambda + a) - (-a)(-a) && (107)\\
&= \lambda^2 - 2a^2 && (108)\\
\lambda &= \pm a\sqrt{2} && (109)
\end{aligned}
$$

Now we find the eigenvectors associated with each of these eigenvalues. First, for $\lambda = +a\sqrt{2}$:

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{2} - 1\right) a & -a \\ -a & \left(\sqrt{2} + 1\right) a \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{110}
$$

This represents two equations, which are equivalent (since the determinant is not invertible, the two equations are not independent). As usual, we just determine the eigenvector to within a constant by setting one of its elements to a constant. Taking $\alpha_1 = A$, and using the equation from the first line:

$$
\begin{aligned}
0 &= \left(\sqrt{2} - 1\right) aA - a\alpha_2 && (111)\\
\alpha_2 &= \left(\sqrt{2} - 1\right) A && (112)
\end{aligned}
$$

Now we normalize the vector $\binom{\alpha_1}{\alpha_2}$:

$$
\begin{aligned}
\left|E = +a\sqrt{2}\right\rangle &= \frac{1}{\sqrt{A^2 + A^2(\sqrt{2} - 1)^2}} \begin{pmatrix} A \\ A(\sqrt{2} - 1) \end{pmatrix} && (113)\\
&= \frac{A}{A\sqrt{1 + 2 - 2\sqrt{2} + 1}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} && (114)\\
&= \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} && (115)\\
&= \frac{1}{\sqrt{2^2 - 2^{3/2}}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} && (116)\\
&= \frac{1}{2^{3/4}\sqrt{2^{1/2} - 1}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} && (117)\\
&= \frac{1}{2^{3/4}} \begin{pmatrix} (\sqrt{2} - 1)^{-1/2} \\ (\sqrt{2} - 1)^{1/2} \end{pmatrix} && (118)
\end{aligned}
$$

Now let's do the same for the negative eigenvalue:

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (-\sqrt{2} - 1)a & -a \\ -a & (-\sqrt{2} + 1)a \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{119}
$$

Again, we set $\alpha_1 = A$ and use the first equation to obtain

$$
\alpha_2 = \left(-\sqrt{2} - 1\right) A \tag{120}
$$

Normalizing the eigenvector, we find:

$$\left|E=-a\sqrt{2}\right\rangle = \frac{A}{\sqrt{A^2+A^2(-\sqrt{2}-1)^2}}\begin{pmatrix}1\\ -\sqrt{2}-1\end{pmatrix} \tag{121}$$

$$= 2^{-3/4}\begin{pmatrix}-(\sqrt{2}+1)^{-1/2}\\ (\sqrt{2}+1)^{1/2}\end{pmatrix} \tag{122}$$

Writing these as linear combinations of the states $|1\rangle$ and $|2\rangle$:

$$\left|E=a\sqrt{2}\right\rangle = 2^{-3/4}\left((\sqrt{2}-1)^{-1/2}|1\rangle + (\sqrt{2}-1)^{1/2}|2\rangle\right) \tag{123}$$

$$\left|E=-a\sqrt{2}\right\rangle = 2^{-3/4}\left(-(\sqrt{2}+1)^{-1/2}|1\rangle + (\sqrt{2}+1)^{1/2}|2\rangle\right) \tag{124}$$